Quantized amplitudes in a nonlinear resonant electrical circuit

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***Abstract*— We present a simple nonlinear resonant analog circuit which demonstrates quantization of resonating amplitudes, for a given excitation level. The system is a simple RLC resonator where C is an active capacitor whose value is related to the current in the circuit. The excitation voltage, synchronously switched at the current frequency, enables electrical supply while keeping the oscillation of the system. The excitation frequency is set to high harmonic of the fundamental oscillation so that anisochronicity can keep constant the amplitude of the circuit voltage and current. The behavior of the circuit is unusual: measured stable amplitudes depend on initial conditions and excitation frequency, for the same amplitude of the excitation. Moreover, a variation of the dumping does not affect significantly the amplitudes as long as the oscillation is observed. Lastly, electrical pulses can change, as in quantum systems, the operating amplitude which is auto-stable without disturbances. Many applications of this circuit can be imagined in microelectronics (including computing), in energy conversion and in time and frequency domains.**

I. INTRODUCTION

Almost all real systems are nonlinear and it is well known that nonlinearity needs complex analysis. Dynamics of nonlinear system can yield to chaotic behavior but, depending on the system and its excitation, stable cycles can be observed. These stable cycles can have different energies and the levels are quantized on the macroscale. Historically, the first studied nonlinear systems where based on mechanics. The basic example taken for nonlinear oscillator has been the pendulum which served as time reference at low oscillation level, i.e. in approximate linear regime, for about three centuries [1]. Obviously, the oscillation of any non linear system can be expressed as a sum of harmonic (Fourier series) oscillations. Two main effects are encountered in non linear oscillating system: the foldover effect which has got its name from the bending of the resonance curve peak in the amplitude versus frequency plot and, more interesting for the proposed topic, superharmonic resonance is simply the resonance with one of this higher harmonics of a nonlinear oscillation. The behaviors of the nonlinear systems have extensively been studied [2-8] on both theoretical and experimental levels. Specific studies of time and frequency applications have shown original results [9-10]. In this work we study the effect of specific excitation with a gated signal. Another approach should also be considered: in non linear systems, chaos behavior is often encountered, but in the case of high Q-factor and sine excitation, synchronous oscillations give stable amplitudes which are sensitive dependent on initial conditions.

In this paper, we describe the association of nonlinear resonant electrical circuit and synchronous excitation which enables different stable amplitude relatively independent on the excitation voltage. In nonlinear circuits, due to huge foldover effect (non bijective amplitude versus frequency curve yields to bistability and hysteresis), two stable amplitudes can be obtained for the same excitation frequency which is close to the fundamental frequency. In our setup the excitation is at odd harmonics of the natural oscillation frequency and the signal is gated, which is unconventional. The actual circuit is formally the equivalent of the Doubochinsky's pendulum [11] which has been excited with an electromagnetic coil, but the use of high frequency and simply adjustable parameters gave quickly various oscillation configurations and a good overview of such a system (with a pendulum having a Q-factor about 200 and 1 second period, more than 10 minutes are needed to obtain a good stabilization of the oscillation). We have designed a completely analog circuit which can be easily understandable and modeled. This circuit is a simple RLC resonator where C is an active capacitor whose value is related to the current in the circuit. A gated driving has been included in order to synchronously excite the oscillation. Some potentiometers have been inserted to allow the preset of the values of the main parameters (gain, gate width, non linearity). The frequency has been chosen about 1 kHz to enable real time response. Obtained results have demonstrated quantized amplitude, even at first odd harmonics of the natural frequency of the oscillation, depending on the initial conditions provided with a low frequency generator. We report some results demonstrating the interest of the concept for self-stabilized devices on the macro- or microscale. Some potential applications are given.

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II. PHYSICAL BASIS

Most phenomena are profitably studied as linear approximations to the real models, mainly because nonlinear models need a very good knowledge of the mathematics, and in the case of oscillating systems, of the nonlinear differential equations which cannot be solved analytically. Moreover, many nonlinear behaviors are not completely known and only approximately modeled. Lastly, for numerous applications, nonlinearity is avoided: this is the case for springs in mechanics, for oscillators in clocks (in this case the nonlinearity is fundamental but it is usually low enough) and for current and voltages in electrical circuits.

We have based this study on nonlinear oscillating circuits. The most well-know nonlinear basic equation is the one of the pendulum:

$$\frac{d^2\theta}{dt^2}+\beta\frac{d\theta}{dt}+\omega_0^2\sin\theta=0 \qquad (1)$$

where θ is the instantaneous angle of the pendulum, β the damping (usually from air drag), and t the time.

For low values of β, the period of the motion can be approximated by using elliptic integral of first kind K(x) [12]. The relative period is expressed as:

$$\frac{T}{T_0}\approx\frac{2}{\pi}K\left[\sin\left(\frac{\theta_0}{2}\right)\right] \qquad (2)$$

where T_0 is the period of the pendulum swinging through a small angle elongation θ_0; the Jacobian elliptical function K(x) has been tabulated [13]. For instance, for 90° angle, $T/T_0 \approx 1.18$. The period of the pendulum clearly depends on the amplitude and this anisochronous behavior enables to stabilize the amplitude of the pendulum excited with a sine force. We have used this feature in our system, but we had a specific excitation in order to synchronize the system and to keep constant the swinging elongation. Concretely, our serial RLC circuit can be modeled by the following equation:

$$L\frac{\partial^2 i}{\partial t^2}+R_S\frac{\partial i}{\partial t}+\frac{i}{C}=A.g(i)\sin(\omega t) \qquad (3)$$

where i represents the electrical current, L the inductance, R_S the serial resistance of the circuit (representing the losses), C_0 the capacitance, A a constant, ω the angular frequency of the excitation and g(i) a gate function so that:

$$g(i)=1 \text{ if } |i|\le|i_0| \text{ and } g(i)=0 \text{ if } |i|>|i_0| \qquad (4)$$

This is obviously the equation of a standard RLC circuit. The only difference is the gated excitation. In order to approach the behavior of the pendulum, we have used analog computing to obtain the appropriate value of the capacitance:

$$\mathrm{C}=C_0\frac{i}{\sin i} \qquad (5)$$

where C_0 is constant. The equation (4) is expressed:

$$\frac{\partial^2 i}{\partial t^2}+\frac{R_S}{L}\frac{\partial i}{\partial t}+\omega_0^2\sin(i)=\frac{A}{L}g(i)\sin(\omega t) \quad \text{with } LC_0\omega_0^2=1 \qquad (6)$$

The solution of this equation cannot be completely obtained by analytic calculations because the excitation term of the equation is a gated signal at angular frequency ω which is usually a high harmonic of the natural angular frequency ω0 of the RLC circuit oscillating at low amplitude.

An approximate value of the amplitudes of oscillation has been given in [11] by using averaging method:

$$a_n\approx B\sqrt{8\left(1-\left(\frac{\omega}{n\omega_0}\right)^2\right)};\text{ with } n=2m+1,\ m=0,1,2,\ldots \qquad (7)$$

where B is a constant. A phenomenological interpretation can be easily given. If we consider the frequency response of nonlinear systems, the possible shapes are shown in Fig. 1. If the system is linear or quasi linear, the conventional resonance curve is obtained. The curve 3 corresponds to not conventional electrical or mechanical systems and will not be discussed here. For most non linear conventional oscillators, the potential well has a cosine shape (for the pendulum or our circuit, the potential energy is expressed: E(x)=1-cos(x), x being the variable) and the response curve is folded on the left side (curve 2). In the frequency range between the two arrows, the hysteretic and bistability effects are obvious; the foldover effect and superharmonic resonance result from the shape of this curve. Experimentally, this means that two different amplitudes can be obtained for the same frequency. Moreover, as the amplitude of oscillation gets larger the period gets longer (frequency decreases); this is a fundamental effect of nonlinearity in curve 2. We have to notice that this behavior is different from the result given by Eq. (7), where the excitation frequency is not the oscillation frequency. Equation (7) shows many possible stable amplitudes corresponding to different oscillation frequencies for the same excitation frequency which is an odd harmonic of f0, the basic frequency of oscillation of the system. The odd numbers are explained by the sign of the excitation current which depends on the sign of the current in the RLC circuit, i.e. on each half-cycle.

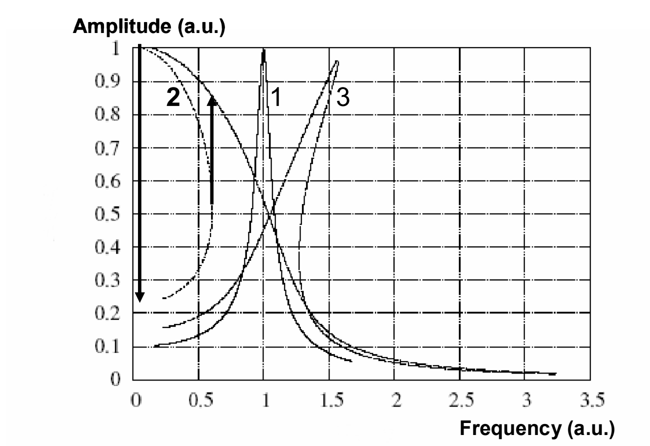


Figure 1. Resonance shape of oscillating system depending of the nonlinearity.

III. PRINCIPLE OF THE NONLINEAR RLC CIRCUIT

In order to correctly design this analog circuit, we have first given a functional scheme separating the different blocks. A diagram of the system is drawn in Fig.2. The inductance is capital for the circuit because of its Q factor. Our 100 mH inductance uses a ferrite pot core. Measured Q factor of the inductance is about 100 at 1 kHz. In order to sense the current, a small (0.5 Ω) resistance R has been inserted in the circuit. A differential amplifier directly gives the appropriate voltage proportional to the current in the circuit. A comparator drives the switch which enables to open the input for the low frequency generator. The value of the reference current i_0 can be preset with a potentiometer. The switch is opened only if $|i| \le |i_0|$. A non linear circuit generates the function of Eq. (5). It is based on conventional wave shaper circuits using resistors and diodes (Piecewise linear approximation).

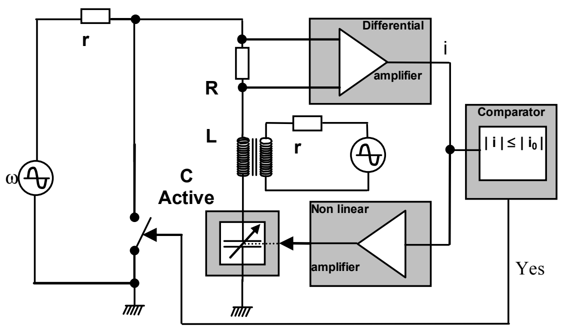


Figure 2. Functional diagram of the non linear circuit

The voltage controlled capacitance is classically based on two operational amplifiers which are used in a gyrator configuration. The maximum variation of the capacity is 1.8, that gives the equivalent variation of the period corresponding to 150° amplitude θ_0 for a simple pendulum.

IV. EXPERIMENTAL RESULTS

The circuit has been supplied and tested in different configurations. We have first checked the non linear behavior of the resonance. In order to easily drive the RLC resonator, we have added a second coil to the inductance so that it acts as a simple transformer and enables the preset of the oscillations. With the chosen coil and capacitance (measured value about 0.38μF for low signal levels), the resonant frequency has been found to be 819Hz, i.e. close to the expected value (1kHz). The standard excitation has been shunted in order to assure the operation of the RLC circuit only, without closed loop effect. The bending of the resonance curve peak in the amplitude versus frequency plot is clearly visible in Fig. 3 where the response is plotted for different levels of the excitation (the excitation has been done through the secondary coil of the inductance). At these levels, and for the measured Q-factor (about 10, depending on the amplitude), the foldover effect is not a source of bistability as shown in the figure (for the highest amplitudes, the curves have been plotted up and down to check the hysteretic effect and bistability). Consequently, it is obvious that any bistability of the circuit originates in another effect. As expected, the nonlinearity decreases the resonant frequencies as the amplitude increases.

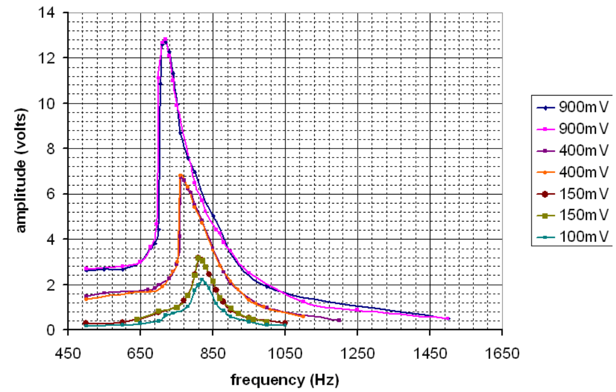
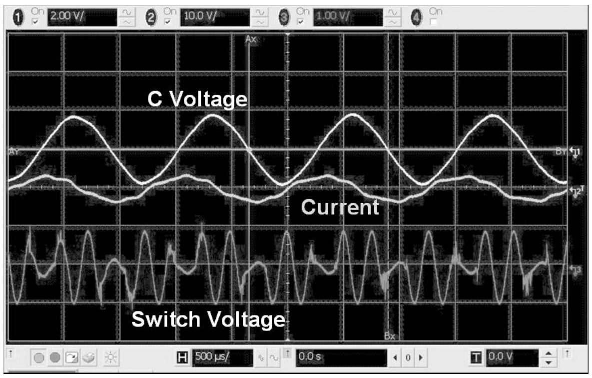


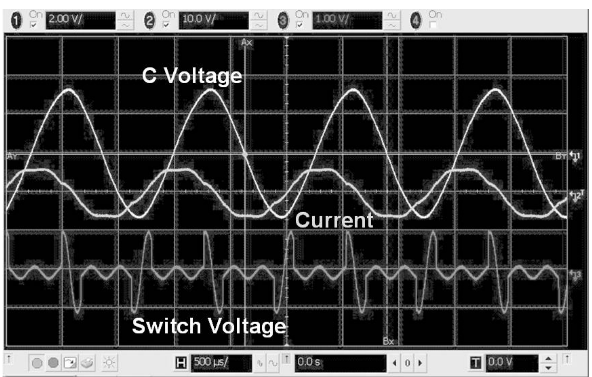
Figure 3. Frequency response of RLC circuit without switching for different values of excitation voltages

In a second step, we have kept the same generator and added another one to understand the operation of the closed-loop circuit. We have used a second generator for the excitation, at a frequency that is an odd harmonic of the oscillation. This configuration has allowed us to preset the different potentiometers and to well understand the operation of the circuit. The first experiments have demonstrated the effect of the gate width that should be correctly preset in order to permit the immediate oscillation of the system when the initial conditions are correctly chosen. An AGILENT (model Infinium DSO 80204B) oscilloscope has been used to record the different signals that are interesting for the understanding of the operation. The first experiments also led us to add a LF power amplifier to assure the oscillation of the circuit when excited through the analog switch. After the necessary preset, we have driven the circuit in the expected excitation mode (odd harmonic of the "natural" frequency of the RLC circuit; in practice, we have checked for different harmonics above $5f_0$). For the first start, we have connected the second generator to the secondary coil of the inductance. Depending on the amplitude level, when we stop it, different oscillation levels can be observed. A long analysis of the oscillogram has shown that the oscillation is kept perfectly stable when no external perturbation is applied. If the oscillation is stopped, the amplitude is kept null and a new pulse is necessary to restart the oscillation. Moreover, we have verified that the excitation amplitude weakly affects the oscillation as long as it is high enough to compensate the energy losses (equivalent loss resistor of the RLC circuit). With our low Q-factor (about 10), typical variation of the amplitude are a few % when the excitation amplitude is doubled; this result is clearly explained by the variation of the oscillation phase lag that tends to keep the same value of the averaged gated signal. The more significant result is the memory effect related to the multi-stability effect. It is demonstrated in Figure 4 where the excitation voltage and frequency is kept constant.

The frequency has been chosen a little bit below $5f_0$ because the frequency decreases at high amplitude levels: exactly 3940 Hz. Depending on the initial conditions, the circuit is oscillating with the amplitude observed in Fig.4.a or 4.b (or null if no pulse is applied to start the oscillation). One may notice that the number of half cycles needed for the excitation is odd, as expected, and is different in the two configurations. The gate function is exactly the same (the opening of the switch is related to the value of the current: the lowest the current, the highest the opening time of the gate function. This result is significant of the non linear behavior of the system: first, obviously, the two oscillation frequencies are different, in accordance with the theory (respectively 805 and 788 Hz, the last value corresponding to the highest amplitude). Secondly, the oscillation is kept on the same level, even if the voltage of the excitation is reasonably varied. Conclusion and prospects



a.



b.

Figure 4. Different signals obtained for the same excitation voltage and frequency, depending on initial conditions

We have demonstrated that multiple stable amplitudes can be easily obtained with a simple nonlinear RLC circuit driven through a synchronous switch at frequencies harmonic of the natural frequency of the resonator. Different interesting features have been demonstrated for a relatively low Q-factor (about 10) and below 1 kHz resonant frequency:

a. the circuit should be driven with odd harmonics so that the average current of each pulse of the gated signal is not null and of the sign of the current

b. the circuit is not self-oscillating when driven with a sine source and the amplitude depends on the initial conditions (energy in the circuit at starting time) for a given amplitude and excitation frequency. Different amplitudes are observed and the corresponding frequency can be explained by anisochronicity and not by the simple bistability related to the "foldover" effect.

c. the amplitude is mainly related to the excitation frequency and the increase of the excitation amplitude weakly affects the stability of the oscillation as long as the oscillation mode is kept; the circuit compensates the excitation amplitude by changing the phase of the gated signal so that the averaged value is kept constant

d. the time width of the gated signal should be optimized to obtain odd number of half-cycles of the excitation signal in a pulse

e. the circuit acts correctly as a frequency divider for different shapes of the non linearity (equivalent to the variation of the shape of the single-well potential); this demonstrates the robustness of the concept.

The presented principle could firstly be used for high efficiency self-stable supplies driven with high frequencies. In the field of memories, the amplitude could be considered as the memory level and multi-state memories become natural (the high level of integration will allow small and low cost systems). But many other applications are opened in the field of time and frequency but also in the field of energy: down-frequency conversion, high efficiency ac to ac (or dc by using rectifiers) converters,…

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